NAMIBIA UNIVERSITY
OF SCIENCE AND TECHNOLOGY

FACULTY OF HEALTH, NATURAL RESOURCES AND APPLIED SCIENCES
SCHOOL OF NATURAL AND APPLIED SCIENCES
DEPARTMENT OF MATHEMATICS, STATISTICS AND ACTUARIAL SCIENCE

QUALIFICATION: BACHELOR OF SCIENCE IN APPLIED MATHEMATICS AND STATISTICS	
QUALIFICATION CODE: 07BSAM	**LEVEL:** 7
COURSE CODE: NUM701S	**COURSE NAME:** NUMERICAL METHODS 1
SESSION: JUNE 2023	**PAPER:** THEORY
DURATION: 3 HOURS	**MARKS:** 100

FIRST OPPORTUNITY EXAMINATION QUESTION PAPER	
EXAMINERS	Dr S. N. NEOSSI NGUETCHUE AND G. S. MBOKOMA
MODERATOR:	Prof S. S. MOTSA

INSTRUCTIONS
1. Answer ALL the questions in the booklet provided.
2. Show clearly all the steps used in the calculations. All numerical results must be given using 4 decimals where necessary unless mentioned otherwise.
3. All written work must be done in blue or black ink and sketches must be done in pencil.

PERMISSIBLE MATERIALS

1. Non-programmable calculator without a cover.

THIS QUESTION PAPER CONSISTS OF 3 PAGES (Including this front page)

Attachments

None

Problem 1 [28 marks]

1-1. Write down the general formula of the Taylor's expansion with the Lagrange and the the integral remainder term respectively of a function $f(x)$ about a point $x = x_0$. [6]

1-2. We want to generate the Taylor series of $f(x) = \sin(x)$ about $x_0 = 0$ in summation form.
1-2-1 Compute f' and f'' and show by induction on $k \in \mathbb{N}$ that [5]

$$f^{(2k+1)}(x) = (-1)^k \cos(x)$$

1-2-2 Deduce the expression of the Taylor series of $f(x) = \sin(x)$ about $x_0 = 0$. [5]

1-3. Suppose that $g : [a,b] \to [a,b]$ is continuous on the real interval $[a,b]$ and is a contraction in the sense that there exists a constant $\lambda \in (0,1)$ such that

$$|g(x) - g(y)| \leq \lambda |x - y|, \text{ for all } x, y \in [a,b].$$

Prove that there exists a unique fixed point in $[a,b]$ and that the fixed point iteration $x_{n+1} = g(x_n)$ converges to it for any $x_0 \in [a,b]$. Also, prove that the error is reduced by a factor of at least λ from each iteration to the next. [12]

Problem 2. [45 marks]

2-1. Write down in details the formulae of the Lagrange and Newton's form of the polynomial that interpolates the set of data points $(x_0, f(x_0)), (x_1, f(x_1)), \ldots, (x_n, f(x_n))$. [7]

2-2. Use the results in **2-1.** to determine the Lagrange and Newton's form of the polynomial that interpolates the set of data points $(1,1)$, $(2,5)$ and $(3,15)$. [18]

2-3. Determine the error term for the formula [15]

$$f'''(x) \approx \frac{1}{2h^3}[3f(x+h) - 10f(x) + 12f(x-h) - 6f(x-2h) + f(x-3h)]$$

2-4. State the central difference formula to approximate $f''(x_0)$ and use it to approximate $f''(0.5)$ when $f(x) = \ln(1+x)$ and $h = 0.001$. [5]

Problem 3. [27 marks]

The fourth-order Runge-Kutta (RK4) method to solve the IVP $y'(t) = f(t,y),\ y(t_0) = y_0$ using n steps is described by the following algorithm

Given f, t_0, y_0, t_f, n, let $h = (t_f - t_0)/n$
For $k = 0, 1, \ldots, n-1$
  $K1 = f(t_k, y_k)$
  $K2 = f(t_k + \frac{h}{2}, y_k + \frac{h}{2}K1)$
  $K3 = f(t_k + \frac{h}{2}, y_k + \frac{h}{2}K2)$
  $K4 = f(t_k + h, y_k + hK3)$
  $y_{k+1} = y_k + (h/6)[K1 + 2K2 + 2K3 + K4]$
  $t_{k+1} = t_k + h$
End For

3-1. Write down the RK4 algorithm for the following specific problem after n steps [7]

$$y'(t) = y - t^2 + 1, \ \ y(0) = 2$$

3-2. In the kingdom of Bana, king Happi The First asked one of his subjects, a prominent mathematician to solve the above IVP using the fourth-order Runge-Kutta (RK4) method. He displayed the results in the form of the following table and purposely skipped some entries.

k	t_k	k_1	k_2	k_3	k_4	y_k
1	0.08	3.0	3.11840		3.24345	2.24969
2	0.16		3.36502		3.49368	
3		3.49351	3.61885			2.80885
4		3.75125		3.88567	4.01730	
5	0.4		4.15061		4.29200	

Compute **only the missing values** by the means of the given ones (don't re-compute them!!). [20]

TOTAL: 100 marks

God bless you !!!